# Linear Algebra I

## 18/12/2017, Monday, 15:00 – 17:00

**1** $(2+7+3+5+3 = 20 \text{ pts})$ **Linear systems of equations**

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Consider the following linear system of equations in the unknowns $x$, $y$, and $z$:

$$\begin{aligned} ax + y + z &= 1 \\ x + ay + z &= 1 \\ x + y + az &= 1. \end{aligned}$$

(a) Write down the augmented matrix.

(b) By performing elementary row operations, put the augmented matrix into row echelon form.

(c) Determine all values of $a$ so that the system is inconsistent.

(d) Determine all values of $a$ so that the system is consistent and find the solution set for such values of $a$.

(e) Determine all values of $a$ so that the system has a unique solution.

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Required Knowledge: **Gauss-elimination, row operations, row echelon form, notions of lead/free variables.**

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Solution:

**1a:** Augmented matrix is given by:

$$\left[\begin{array}{ccc|c} a & 1 & 1 & 1 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \end{array}\right].$$

**1b:**

$$\left[\begin{array}{ccc|c} a & 1 & 1 & 1 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \end{array}\right] \xrightarrow[\textcircled{2} = \textcircled{1}]{\textcircled{1} = \textcircled{2}} \left[\begin{array}{ccc|c} 1 & a & 1 & 1 \\ a & 1 & 1 & 1 \\ 1 & 1 & a & 1 \end{array}\right] \xrightarrow[\textcircled{3} = \textcircled{3} - \textcircled{1}]{\textcircled{2} = \textcircled{2} - a \cdot \textcircled{1}} \left[\begin{array}{ccc|c} 1 & a & 1 & 1 \\ 0 & 1-a^2 & 1-a & 1-a \\ 0 & 1-a & a-1 & 0 \end{array}\right]$$

We can distinguish two cases depending on the value of $a$:

**Case 1: $a = 1$**

In this case, the matrix we obtained in the previous step is already in row echelon form:

$$\left[\begin{array}{ccc|c} 1 & a & 1 & 1 \\ 0 & 1-a^2 & 1-a & 1-a \\ 0 & 1-a & a-1 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

**Case 2: $a \neq 1$**

$$\left[\begin{array}{ccc|c} 1 & a & 1 & 1 \\ 0 & 1-a^2 & 1-a & 1-a \\ 0 & 1-a & a-1 & 0 \end{array}\right] \xrightarrow[\textcircled{3} = \textcircled{2}]{\textcircled{2} = \textcircled{3}} \left[\begin{array}{ccc|c} 1 & a & 1 & 1 \\ 0 & 1-a & a-1 & 0 \\ 0 & 1-a^2 & 1-a & 1-a \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & a & 1 & 1 \\ 0 & 1-a & a-1 & 0 \\ 0 & 1-a^2 & 1-a & 1-a \end{array}\right] \xrightarrow{\textcircled{2} = \frac{1}{1-a} \cdot \textcircled{2}} \left[\begin{array}{ccc|c} 1 & a & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1-a^2 & 1-a & 1-a \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & a & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1-a^2 & 1-a & 1-a \end{array}\right] \xrightarrow{\textcircled{3} = \textcircled{3} + (a^2-1) \cdot \textcircled{2}} \left[\begin{array}{ccc|c} 1 & a & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 2-a-a^2 & 1-a \end{array}\right]$$

Note that $a^2 + a - 2 = 0$ if and only if $a = 1$ or $a = -2$. Since we have already assumed that $a \neq 1$. The term $2 - a - a^2$ can be zero only if $a = -2$. This leads to two subcases.

**Case 2.1: $a \neq 1$ and $a = -2$**

In this case, we have

$$\left[\begin{array}{ccc|c} 1 & a & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 2-a-a^2 & 1-a \end{array}\right] \xrightarrow{\textcircled{3} = \frac{1}{3} \cdot \textcircled{3}} \left[\begin{array}{ccc|c} 1 & a & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

**Case 2.2: $a \neq 1$ and $a \neq -2$**

$$\left[\begin{array}{ccc|c} 1 & a & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 2-a-a^2 & 1-a \end{array}\right] \xrightarrow{\textcircled{3} = \frac{1}{2-a-a^2} \cdot \textcircled{3}} \left[\begin{array}{ccc|c} 1 & a & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & \frac{1}{a+2} \end{array}\right]$$

**1c:** We have obtained the following row echelon forms:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \text{if } a = 1$$

$$\left[\begin{array}{ccc|c} 1 & a & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \qquad \text{if } a \neq 1 \text{ and } a = -2$$

$$\left[\begin{array}{ccc|c} 1 & a & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & \frac{1}{a+2} \end{array}\right] \qquad \text{if } a \neq 1 \text{ and } a \neq -2.$$

Therefore, we see that the system is inconsistent if and only if $a \neq 1$ and $a = -2$.
**1d:** The system is consistent if and only if $(a = 1)$ or $(a \neq 1$ and $a \neq -2)$.

If $a = 1$, then $x$ is the lead variable and $y, z$ are free variables. This leads to the general solution:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1-y-z \\ y \\ z \end{bmatrix}.$$

If $a \neq 1$ and $a \neq -2$, then $x, y, z$ are all lead variables. This leads to the general solution:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{a+2} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

**1e:** From the previous subproblem, we see that the system has a unique solution if and only if $a \neq 1$ and $a \neq -2$.

Find all values of $a$, $b$, $c$, $d$, $e$, and $f$ such that the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ a & b & b & b \\ a & c & d & d \\ a & c & e & f \end{bmatrix}$$

is singular.

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Required Knowledge: **Determinants, nonsingular matrices.**

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Solution:

First we compute the determinant. By applying row operations and cofactor expansions, we obtain:

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & b & b \\ a & c & d & d \\ a & c & e & f \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & b-a & b-a & b-a \\ 0 & c-a & d-a & d-a \\ 0 & c-a & e-a & f-a \end{vmatrix} \quad \left\{\begin{matrix} \text{row operations} \\ ② = ② - a \cdot ① \\ ③ = ③ - a \cdot ① \\ ④ = ④ - a \cdot ① \end{matrix}\right\}$$

$$= \begin{vmatrix} b-a & b-a & b-a \\ c-a & d-a & d-a \\ c-a & e-a & f-a \end{vmatrix} \quad \left\{\begin{matrix} \text{cofactor expansion} \\ \text{with respect to} \\ \text{column ①} \end{matrix}\right\}$$

$$= (b-a)\begin{vmatrix} 1 & 1 & 1 \\ c-a & d-a & d-a \\ c-a & e-a & f-a \end{vmatrix} = (b-a)\begin{vmatrix} 1 & 1 & 1 \\ 0 & d-c & d-c \\ 0 & e-c & f-c \end{vmatrix} \quad \left\{\begin{matrix} \text{row operations} \\ ② = ② - (c-a) \cdot ① \\ ③ = ③ - (c-a) \cdot ① \end{matrix}\right\}$$

$$= (b-a)\begin{vmatrix} d-c & d-c \\ e-c & f-c \end{vmatrix} \quad \left\{\begin{matrix} \text{cofactor expansion} \\ \text{with respect to} \\ \text{column ①} \end{matrix}\right\}$$

$$= (b-a)(d-c)\begin{vmatrix} 1 & 1 \\ e-c & f-c \end{vmatrix}$$

$$= (b-a)(d-c)(f-c-e+c) = (b-a)(d-c)(f-e).$$

A square matrix is singular if and only if its determinant is zero. Therefore, the matrix given in this problem is singular if and only if $a = b$ or $c = d$ or $e = f$.

Let $A$ and $B$ be $n \times n$ matrices. Suppose that $A$ is nonsingular.

(a) Show that the matrix

$$M = \begin{bmatrix} A & B \\ B & A \end{bmatrix}$$

is nonsingular if and only if the matrix $A - BA^{-1}B$ is nonsingular.

(b) Suppose that $A - BA^{-1}B$ is nonsingular and find the inverse of $M$.

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Required Knowledge: **Partitioned matrices, nonsingular matrices, and inverse.**

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Solution:

**3a:** 'only if': Suppose that the matrix

$$M = \begin{bmatrix} A & B \\ B & A \end{bmatrix}$$

is nonsingular. Let $x \in \mathbb{R}^n$ be such that $(A - BA^{-1}B)x = 0$. It is enough to show that $x = 0_n$. Note that

$$M \begin{bmatrix} -A^{-1}Bx \\ x \end{bmatrix} = \begin{bmatrix} A & B \\ B & A \end{bmatrix} \begin{bmatrix} -A^{-1}Bx \\ x \end{bmatrix} = \begin{bmatrix} -AA^{-1}Bx + Bx \\ -BA^{-1}Bx + Ax \end{bmatrix} = \begin{bmatrix} 0_n \\ 0_n \end{bmatrix}.$$

Since $M$ is nonsingular, we see that $x = 0$. Hence, $A - BA^{-1}B$ is nonsingular.

'if': Suppose that $A - BA^{-1}B$ is nonsingular. Let $x, y \in \mathbb{R}^n$ be such that

$$0_{2n} = M \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} A & B \\ B & A \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} Ax + By \\ Bx + Ay \end{bmatrix}.$$

It is enough to show that $x = y = 0_n$. As such, we have

$$Ax + By = 0_n \tag{1}$$
$$Bx + Ay = 0_n \tag{2}$$

Since $A$ is nonsingular, it follows from (1) that $x = -A^{-1}By$. Then, (2) implies that $(A - BA^{-1}B)x = 0_n$. Since $A - BA^{-1}B$, we see that $x = 0_n$. Then, (2) yields that $Ay = 0_n$. Since $A$ is nonsingular, we get $y = 0_n$. Consequently, $M$ is nonsingular.

**3b:** Let $U, V, W, X$ be $n \times n$ matrices such that

$$M \begin{bmatrix} U & V \\ W & X \end{bmatrix} = \begin{bmatrix} A & B \\ B & A \end{bmatrix} \begin{bmatrix} U & V \\ W & X \end{bmatrix} = \begin{bmatrix} I_n & 0_{n,n} \\ 0_{n,n} & I_n \end{bmatrix}.$$

Therefore, we have

$$AU + BW = I_n \tag{3}$$
$$AV + BX = 0_{n,n} \tag{4}$$
$$BU + AW = 0_{n,n} \tag{5}$$
$$BV + AX = I_n. \tag{6}$$

Since $A$ is nonsingular, we can solve $V$ from (4):

$$V = -A^{-1}BX.$$

Together with (6), this implies that

$$X = (A - BA^{-1}B)^{-1}$$

in view of nonsingularity of $A - BA^{-1}B$ and thus

$$V = -A^{-1}B(A - BA^{-1}B)^{-1}.$$

Similarly, we can solve $W$ from (5) and use it in (3) in order to obtain

$$W = -A^{-1}B(A - BA^{-1}B)^{-1} \quad \text{and} \quad U = (A - BA^{-1}B)^{-1}.$$

Therefore, we get

$$M^{-1} = \begin{bmatrix} (A - BA^{-1}B)^{-1} & -A^{-1}B(A - BA^{-1}B)^{-1} \\ -A^{-1}B(A - BA^{-1}B)^{-1} & (A - BA^{-1}B)^{-1} \end{bmatrix}.$$

An alternative approach would be solving $U$ and $X$ from (3) and (6). This results in

$$U = A^{-1}(I_n - BW) \tag{7}$$

in view of nonsingularity of $A$. Substituting (7) in (5) yields $(A - BA^{-1}B)W = -BA^{-1}$. Since $A - BA^{-1}B$ is nonsingular, we get

$$W = -(A - BA^{-1}B)^{-1}BA^{-1}. \tag{8}$$

As such, (7) results in

$$U = A^{-1} + A^{-1}B(A - BA^{-1}B)^{-1}BA^{-1}.$$

Similarly, we can solve $X$ from (6) and use (5) to obtain

$$V = -(A - BA^{-1}B)^{-1}BA^{-1}$$

and

$$X = A^{-1} + A^{-1}B(A - BA^{-1}B)^{-1}BA^{-1}.$$

Hence, we obtain

$$M^{-1} = \begin{bmatrix} A^{-1} + A^{-1}B(A - BA^{-1}B)^{-1}BA^{-1} & -(A - BA^{-1}B)^{-1}BA^{-1} \\ -(A - BA^{-1}B)^{-1}BA^{-1} & A^{-1} + A^{-1}B(A - BA^{-1}B)^{-1}BA^{-1} \end{bmatrix}.$$

(a) Let $E = (\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n)$ be an ordered basis for the vector space $V$.

(i) Show that the vectors $\boldsymbol{v}_1 + \boldsymbol{v}_2, \boldsymbol{v}_2 + \boldsymbol{v}_3, \ldots, \boldsymbol{v}_{n-1} + \boldsymbol{v}_n, \boldsymbol{v}_n$ form a basis for $V$.

(ii) Find the transition matrix corresponding to the change of basis from $E = (\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n)$ to $F = (\boldsymbol{v}_1 + \boldsymbol{v}_2, \boldsymbol{v}_2 + \boldsymbol{v}_3, \ldots, \boldsymbol{v}_{n-1} + \boldsymbol{v}_n, \boldsymbol{v}_n)$.

(b) Consider the vector space $P_4$. Let

$$S = \{p \in P_4 \mid p(1) = 0 \text{ and } p'(1) = 0\}$$

where $p'(x)$ denotes the derivative of $p(x)$.

(i) Show that $S$ is a subspace.

(ii) Find a basis for $S$ and determine its dimension.

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REQUIRED KNOWLEDGE: **Subspaces, basis, dimension, change of basis.**

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SOLUTION:

**4a(i):** The vectors $\boldsymbol{v}_1 + \boldsymbol{v}_2, \boldsymbol{v}_2 + \boldsymbol{v}_3, \ldots, \boldsymbol{v}_{n-1} + \boldsymbol{v}_n, \boldsymbol{v}_n$ form a basis for $V$ if they span $V$ and are linearly independent. For the former, let $\boldsymbol{x} \in V$. Since $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ form a basis, there exist scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that

$$\boldsymbol{x} = \alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2 + \cdots + \alpha_n \boldsymbol{v}_n. \tag{9}$$

We need to show that there exist scalars $\beta_1, \beta_2, \ldots, \beta_n$ such that

$$\boldsymbol{x} = \beta_1(\boldsymbol{v}_1 + \boldsymbol{v}_2) + \beta_2(\boldsymbol{v}_2 + \boldsymbol{v}_3) + \cdots + \beta_{n-1}(\boldsymbol{v}_{n-1} + \boldsymbol{v}_n) + \beta_n \boldsymbol{v}_n.$$

This would mean that

$$\boldsymbol{x} = \beta_1 \boldsymbol{v}_1 + (\beta_1 + \beta_2)\boldsymbol{v}_2 + (\beta_2 + \beta_3)\boldsymbol{v}_3 + \cdots + (\beta_{n-1} + \beta_n)\boldsymbol{v}_n. \tag{10}$$

By comparing this with (9), we see that

$$\begin{aligned}
\beta_1 &= \alpha_1 \\
\beta_2 &= \alpha_2 - \alpha_1 \\
\beta_3 &= \alpha_3 - \alpha_2 + \alpha_1 \\
&\vdots \\
\beta_n &= \alpha_n - \alpha_{n-1} + - \cdots + (-1)^{(n+1)}\alpha_1.
\end{aligned}$$

Therefore, the vectors $\boldsymbol{v}_1 + \boldsymbol{v}_2, \boldsymbol{v}_2 + \boldsymbol{v}_3, \ldots, \boldsymbol{v}_{n-1} + \boldsymbol{v}_n, \boldsymbol{v}_n$ span $V$.

To show that they are linearly independent, let $c_1, c_2, \ldots, c_n$ be scalars such that

$$c_1(\boldsymbol{v}_1 + \boldsymbol{v}_2) + c_2(\boldsymbol{v}_2 + \boldsymbol{v}_3) + \cdots + c_{n-1}(\boldsymbol{v}_{n-1} + \boldsymbol{v}_n) + c_n \boldsymbol{v}_n = \mathbf{0}.$$

Equivalently, we have

$$c_1 \boldsymbol{v}_1 + (c_1 + c_2)\boldsymbol{v}_2 + (c_2 + c_3)\boldsymbol{v}_3 + \cdots + (c_{n-1} + c_n)\boldsymbol{v}_n = \mathbf{0}$$

It follows from linear independence of the vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ that

$$\begin{aligned}
c_1 &= 0 \\
c_1 + c_2 &= 0 \\
&\vdots \\
c_{n-1} + c_n &= 0.
\end{aligned}$$

As such, we see that $c_1 = c_2 = \cdots = c_n = 0$. Therefore, the vectors $\boldsymbol{v}_1 + \boldsymbol{v}_2, \boldsymbol{v}_2 + \boldsymbol{v}_3, \ldots, \boldsymbol{v}_{n-1} + \boldsymbol{v}_n, \boldsymbol{v}_n$ are linearly independent.

**4a(ii):**

We begin with expressing the basis vectors in $E$ as linear combinations of those in $F$:

$$\begin{aligned}
\boldsymbol{v}_1 &= +1 \cdot (\boldsymbol{v}_1 + \boldsymbol{v}_2) - 1 \cdot (\boldsymbol{v}_2 + \boldsymbol{v}_3) + 1 \cdot (\boldsymbol{v}_3 + \boldsymbol{v}_4) + \cdots + (-1)^n \cdot (\boldsymbol{v}_{n-1} + \boldsymbol{v}_n) + (-1)^{n+1} \cdot \boldsymbol{v}_n \\
\boldsymbol{v}_2 &= +0 \cdot (\boldsymbol{v}_1 + \boldsymbol{v}_2) + 1 \cdot (\boldsymbol{v}_2 + \boldsymbol{v}_3) - 1 \cdot (\boldsymbol{v}_3 + \boldsymbol{v}_4) + \cdots + (-1)^{n-1} \cdot (\boldsymbol{v}_{n-1} + \boldsymbol{v}_n) + (-1)^n \cdot \boldsymbol{v}_n \\
&\vdots \\
\boldsymbol{v}_{n-1} &= +0 \cdot (\boldsymbol{v}_1 + \boldsymbol{v}_2) + 0 \cdot (\boldsymbol{v}_2 + \boldsymbol{v}_3) + 0 \cdot (\boldsymbol{v}_3 + \boldsymbol{v}_4) + \cdots + 1 \cdot (\boldsymbol{v}_{n-1} + \boldsymbol{v}_n) - 1 \cdot \boldsymbol{v}_n \\
\boldsymbol{v}_n &= +0 \cdot (\boldsymbol{v}_1 + \boldsymbol{v}_2) + 0 \cdot (\boldsymbol{v}_2 + \boldsymbol{v}_3) + 0 \cdot (\boldsymbol{v}_3 + \boldsymbol{v}_4) + \cdots + 0 \cdot (\boldsymbol{v}_{n-1} + \boldsymbol{v}_n) + 1 \cdot \boldsymbol{v}_n.
\end{aligned}$$

Therefore, the transition matrix is given by:

$$T = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & -1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ (-1)^n & (-1)^{n-1} & (-1)^{n-2} & \cdots & -1 & 1 & 0 \\ (-1)^{n+1} & (-1)^n & (-1)^{n-1} & \cdots & 1 & -1 & 1 \end{bmatrix}$$

**4b(i):** First, we note that the zero polynomial belongs to $S$. As such, $S_2$ is nonempty.

Let $\alpha$ be a scalar and $p \in S$. Note that

$$(\alpha p)(1) = \alpha p(1) = 0 \text{ and } (\alpha p)'(1) = \alpha p'(1) = 0.$$

This means that $\alpha p \in S$.

Now, let $p, q \in S$. Note that

$$(p + q)(1) = p(1) + q(1) = 0 \text{ and } (p + q)'(1) = p'(1) + q'(1) = 0.$$

As such, $p + q \in S$.

Consequently, $S$ is a subspace.

**4b(ii):** Let $p \in P_4$. Note that if $p(x) = ax^3 + bx^2 + cx + d$ then $p'(x) = 3ax^2 + 2bx + c$. Therefore, $p \in S$ if and only if

$$a + b + c + d = 0 \quad \text{and} \quad 3a + 2b + c = 0.$$

Solving this linear system, we see that $c, d$ are free variables and $a, b$ are lead variables given by

$$a = c + 2d \quad \text{and} \quad b = -2c - 3d.$$

Thus, $p \in S$ if and only if

$$p(x) = (c + 2d)x^3 + (-2c - 3d)x^2 + cx + d = c(x^3 - 2x^2 + x) + d(2x^3 - 3x^2 + 1).$$

for some scalars $c$ and $d$. This means that the vectors $x^3 - 2x^2 + x, 2x^3 - 3x^2 + 1$ span $S$. Therefore, they would form a basis for $S$ if they are linearly independent. To verify linear independence, let $\alpha$ and $\beta$ be such that

$$\alpha(x^3 - 2x^2 + x) + \beta(2x^3 - 3x^2 + 1) = 0.$$

This would mean that

$$\alpha + 2\beta = 0, \quad -2\alpha - 3\beta = 0, \quad \alpha = 0 \quad \text{and} \quad \beta = 0.$$

Clearly, the only solution is $\alpha = \beta = 0$. As such, the vectors $x^3 - 2x^2 + x, 2x^3 - 3x^2 + 1$ are linearly independent and thus form a basis for $S$.

The dimension of a space is the cardinality of its basis. Therefore, $\dim(S) = 2$.